Introduction to trigonometry

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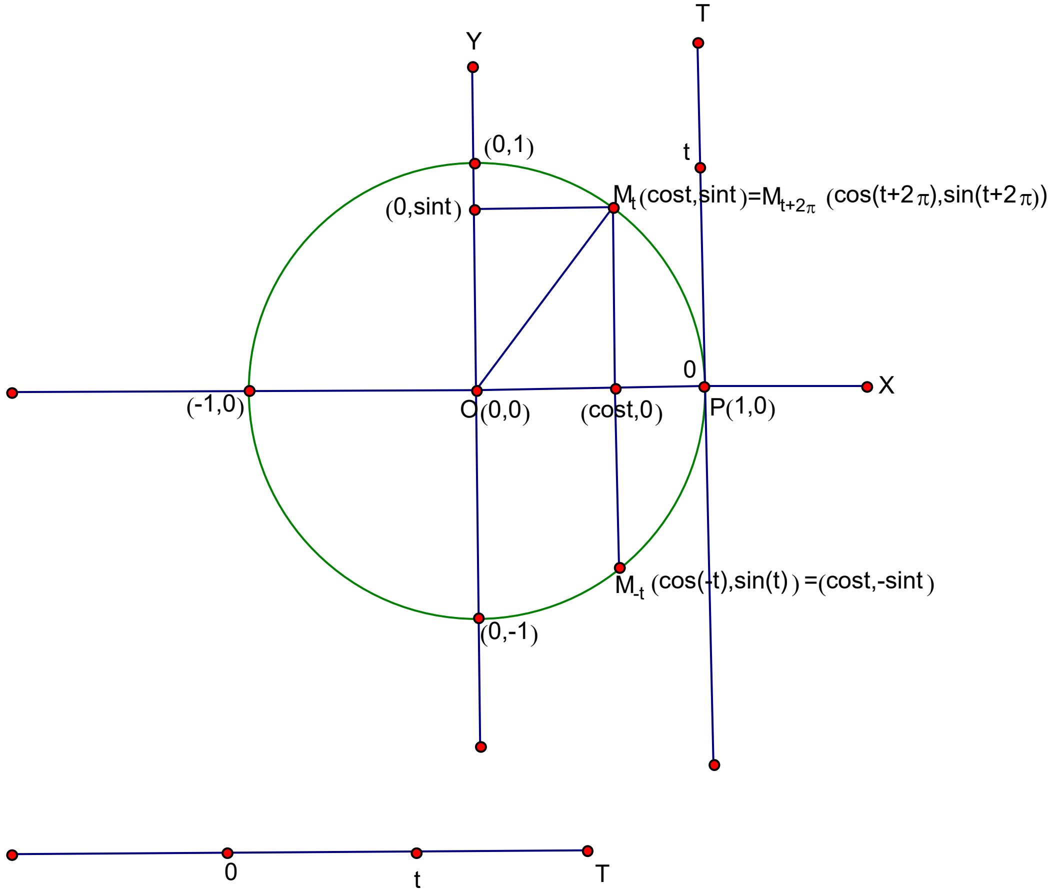
1. Definition of functions $\sin t$, $\cos t$ (for any real t)



Orientation on the plane: Positive-counterclockwise; negative -clockwise.
Let $C_1(O)$ be the circle with radius 1(unite circle) with center in origin O of coordinate system XOY. Let t be a point on numerical line T and let $t \mapsto M_t : T \to C_1(O)$ be the mapping of numerical line T on unite circle $C_1(O)$ defined as follows:
To any number t we set in correspondence point M_t on the circle such that ark PM_t (positively oriented if $t > 0$ and negatively oriented if $t < 0$) has length equal to $|t|$. We will give special names to coordinates of point M_t : α $\cos t$ for first coordinate (x –coordinate) and $\sin t$ for second coordinate (y –coordinate).
That is $M_t(\cos t, \sin t)$ and $\cos t$ is orthogonal projection of M_t on axe OX (vertical projection), $\sin t$ is orthogonal projection of M_t on axe OY (horizontal projection).
Since $M_t(\cos t, \sin t)$ is point on the unite circle then
$\boxed{\cos^2 t + \sin^2 t = 1}$ for any real t. ($\cos^2 t + \sin^2 t = 1$ yield

$\cos^2 t \le 1 \Leftrightarrow |\cos t| \le 1$ and $\sin^2 t \le 1 \Leftrightarrow |\sin t| \le 1$).
Also, from definition, since

$M_{-t}(\cos(-t), \sin(-t)) = (\cos t, -\sin t)$ and $M_{t \pm 2\pi} = M_t$

follows that

$\boxed{\cos(-t) = \cos t}$ (cosine is even function on $\mathbb{R}$),

$\boxed{\sin(-t) = -\sin t}$ (sine is odd function on $\mathbb{R}$)

and $\boxed{\cos(t + 2n\pi) = \cos t, \sin(t + 2n\pi) = \sin t}$

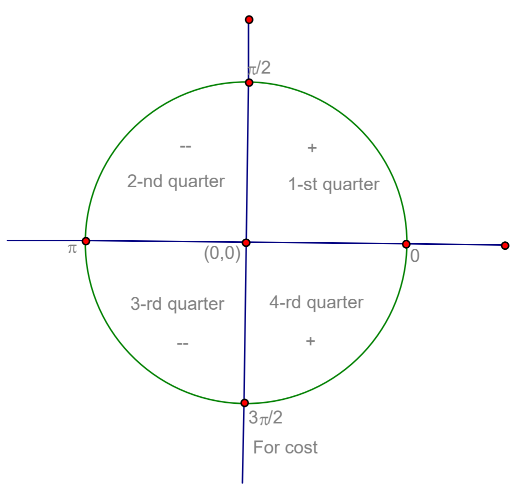
for any real t and any integer n (cosine and sine are 2π –periodical functions on $\mathbb{R}$).

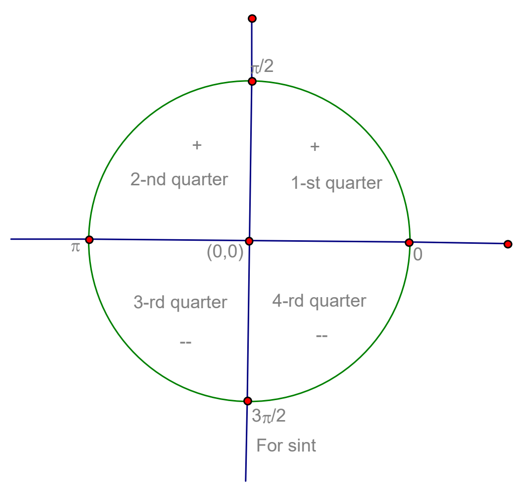
From definition (as we can see on the picture with unite circle) follow that $\sin(\pi - t) = \sin t$ and also we can see that $\begin{cases} \sin t_1 = \sin t_2 \\ t_1, t_2 \in [0, 2\pi) \end{cases} \Leftrightarrow \left[\begin{array}{l} t_1 = t_2 \\ t_1 = \pi - t_2 \end{array} \right.$

and $\begin{cases} \cos t_1 = \cot s_2 \\ t_1, t_2 \in [0, 2\pi) \end{cases} \Leftrightarrow \left[\begin{array}{l} t_1 = t_2 \\ t_1 = -t_2 \end{array} \right.$.

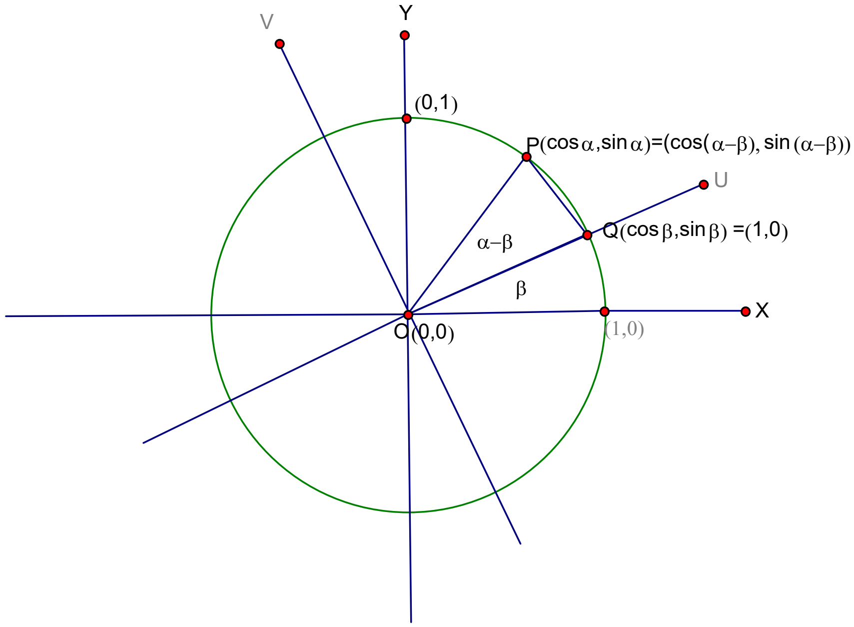
Rigorous (formal) proof will be later.

2. Sign distribution for sine and cosine.





3. Sum and difference formulas for Sine and Cosine.



In XOY coordinate system:

$PQ^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 =$

$\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta =$

$(\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) =$

$2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$.

After rotation in positive orientation axes OX and OY on angle β (new axes OU passed through point Q) we obtain new coordinates system UOV.

Since in coordinate system UOV point P have coordinates $(\cos(\alpha - \beta), \sin(\alpha - \beta))$ and point Q have coordinates $(1,0)$ then

$PQ^2 = (\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta) - 0)^2 =$

$\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta) = 2 - 2\cos(\alpha - \beta)$.

Then we obtain $2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2 - 2\cos(\alpha - \beta) \Leftrightarrow$

(1) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Since $\cos(-\beta) = \cos\beta, \sin(-\beta) = -\sin\beta$ and $\cos(\alpha + \beta) = \cos(\alpha - (-\beta))$ then by replacing β in (**1**) with $-\beta$ we obtain

$\cos(\alpha + \beta) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. So,

(2) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

By replacing (α, β) in (**1**) with $\left(\frac{\pi}{2}, t\right)$ we obtain

$$\cos\left(\frac{\pi}{2} - t\right) = \cos\frac{\pi}{2}\cos t + \sin\frac{\pi}{2}\sin t = \sin t$$

for any real t (because $\cos\frac{\pi}{2} = 0, \sin\frac{\pi}{2} = 1$).

Since $\cos t = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - t\right)\right)$ then by replacing t in equality $\cos\left(\frac{\pi}{2} - t\right) = \sin t$ with $\left(\frac{\pi}{2} - t\right)$ we obtain $\cos t = \sin\left(\frac{\pi}{2} - t\right)$.

So, we have two correlation $\boxed{\cos\left(\frac{\pi}{2} - t\right) = \sin t \text{ and } \sin\left(\frac{\pi}{2} - t\right) = \cos t}$

for any real t.

Using these correlation we obtain

$\sin(\alpha + \beta) = \cos\left(\frac{\pi}{2} - (\alpha + \beta)\right) = \cos\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right) =$

$\cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta =$

$\sin\alpha\cos\beta + \cos\alpha\sin\beta$ and then $\sin(\alpha - \beta) =$

$\sin(\alpha + (-\beta)) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) =$

$\sin\alpha\cos\beta - \cos\alpha\sin\beta.$

Thus, we have another two important formulas for sine:

(**3**) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and

(**4**) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

In particular, applying formulas (**2**) and (**3**) for $\beta = \alpha$ we obtain

(**5**) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ and

(**6**) $\sin 2\alpha = 2\sin\alpha\cos\beta.$

Since $\cos^2\alpha + \sin^2\alpha = 1$ then $1 + \cos 2\alpha = \cos^2\alpha + \sin^2\alpha + \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha$ and $1 - \cos 2\alpha = \cos^2\alpha + \sin^2\alpha - (\cos^2\alpha - \sin^2\alpha) = 2\sin^2\alpha.$

Thus more two formulas:

(**7**) $1 + \cos 2\alpha = 2\cos^2\alpha,$

(**8**) $1 - \cos 2\alpha = 2\sin^2\alpha.$

By replacing α in formulas (**7**) and (**8**) with $\frac{\alpha}{2}$ we get two new formulas which gives opportunity calculate sine and cosine of half argument, that is

(**9**) $\left|\cos\frac{\alpha}{2}\right| = \sqrt{\frac{1 + \cos\alpha}{2}}$ and

(**10**) $\left|\sin\frac{\alpha}{2}\right| = \sqrt{\frac{1 - \cos\alpha}{2}}$.

We can determine sign of $\cos\frac{\alpha}{2}$ and $\sin\frac{\alpha}{2}$ using information about position of $\frac{\alpha}{2}$ on the unite circle, namely $\cos\frac{\alpha}{2} = \sqrt{\frac{1 + \cos\alpha}{2}}$ if end of the arc which correspondent to $\frac{\alpha}{2}$ belong to right hulf-circle, otherwise

$\cos\frac{\alpha}{2} = -\sqrt{\frac{1 + \cos\alpha}{2}}$.

And $\sin\frac{\alpha}{2} = \sqrt{\frac{1 - \cos\alpha}{2}}$ if end of the arc correspondent to $\frac{\alpha}{2}$ belong to upper hulf-circle, otherwise $\sin\frac{\alpha}{2} = -\sqrt{\frac{1 - \cos\alpha}{2}}$.

Also we will note "Reduction formulas" as more applications of formulas (**1**)-(**4**):

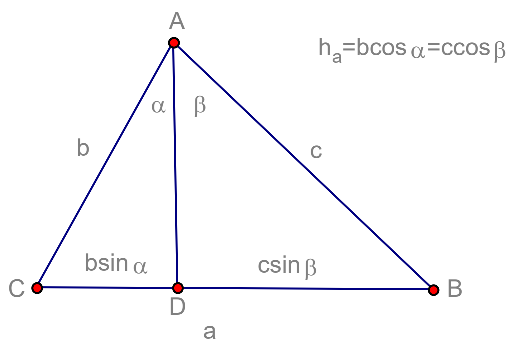
$\sin\left(\frac{\pi}{2} + t\right) = \cos t,$

$\cos\left(\frac{\pi}{2} + t\right) = -\sin t,$

$\sin(\pi - t) = \sin t,$

$\sin(\pi + t) = -\sin t,$
$\cos(\pi \pm t) = -\cos t.$

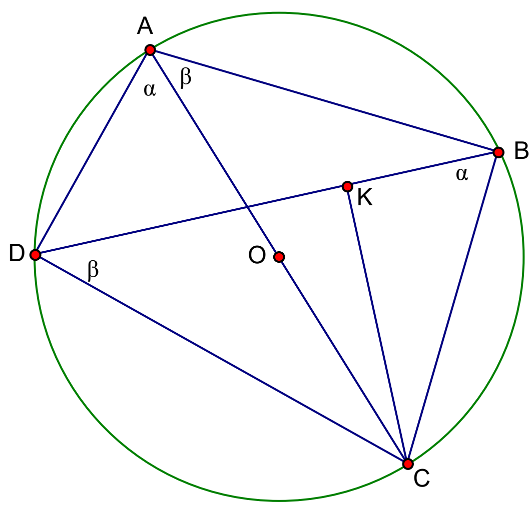
4. Geometric proof of $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.



Let $AD \perp BC$. Then $CD = b\sin\alpha, BD = c\sin\beta,\ AD = b\cos\alpha = c\cos\beta$ and $[ABC] = [ACD] + [ABD] = \dfrac{CD \cdot AD}{2} + \dfrac{BD \cdot AD}{2} = \dfrac{b\sin\alpha \cdot c\cos\beta}{2} + \dfrac{c\sin\beta \cdot b\cos\alpha}{2} = \dfrac{bc\sin\alpha\cos\beta + bc\cos\alpha\sin\beta}{2} = \dfrac{bc(\sin\alpha\cos\beta + \cos\alpha\sin\beta)}{2}$.

From the other hand $[ABC] = \dfrac{bc\sin(\alpha+\beta)}{2}$. Then $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

5. Sine of sum of two angles using Sine-Theorem.

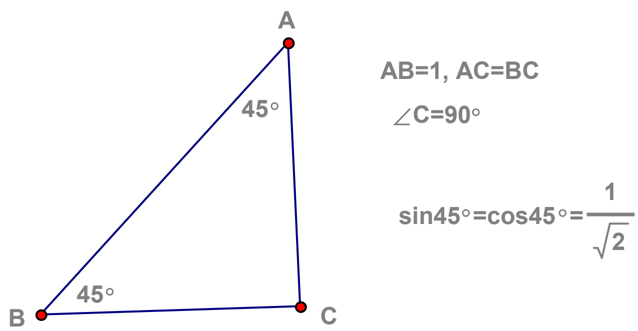


Assume that AC is diameter of the circle and equal 1. Since $\triangle ABC$ and $\triangle ADC$ are right triangles then $DC = \sin\alpha, BC = \sin\beta$. Also note then by Sine-Theorem $DB = \sin(\alpha+\beta)$.

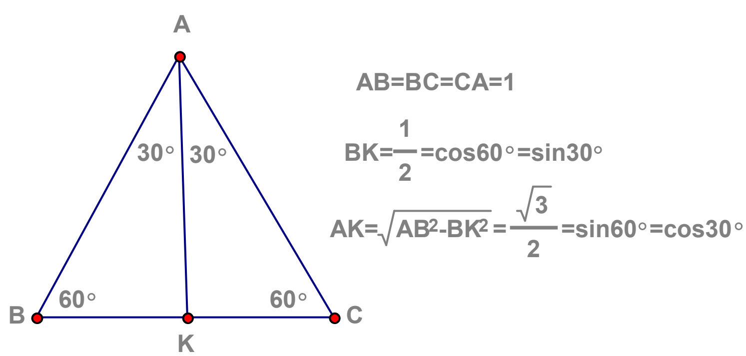
Let $CK \perp DB$ then $DB = DK + KB = DC\cos\beta + BC\cos\gamma = \sin\alpha\cos\beta + \sin\beta\cos\gamma$.

Thus, $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\gamma$.

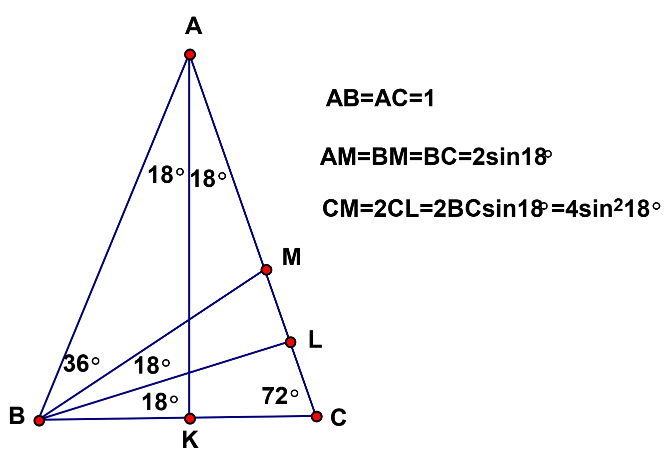
6. Values of sine and cosine for concrete angles.



$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$



$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2}, \sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$



$1 = AM + CM = 2\sin 18^\circ + 4\sin^2 18^\circ$ implies $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$.

Hence, $\cos 18^\circ = AK = \sqrt{1 - BK^2} = \sqrt{1 - \sin^2 18^\circ} =$
$\sqrt{1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2} = \frac{\sqrt{2\sqrt{5} + 10}}{4}$.
(Later we will represent another way (algebraic)
for calculation of $\sin 18^\circ$).

7. Application of formulas (9), (10)
Example. Find $\cos\frac{\pi}{12}$.

$$\left|\cos\frac{\pi}{12}\right| = \sqrt{\frac{1 + \cos\frac{\pi}{6}}{2}} = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 + \sqrt{3}}{4}} = \sqrt{\frac{4 + 2\sqrt{3}}{8}} = \frac{1 + \sqrt{3}}{2\sqrt{2}}.$$

Since $\frac{\pi}{12}$ belong to 1-st quarter then $\cos\frac{\pi}{12} = \frac{1 + \sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2} + \sqrt{6}}{4}$.
Find $\sin\frac{\pi}{12}$
Example.
Find $\cos\frac{5\pi}{12}$
$\cos\frac{5\pi}{12} = \cos\left(\frac{6\pi}{12} - \frac{\pi}{12}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{12}\right) = \sin\frac{\pi}{12}$.
Find $\cos\frac{7\pi}{12}, \sin\frac{11\pi}{12}, \sin\frac{13\pi}{12}$.

8. Product to Sum Formulas.
By addition (**1**) and (**2**) and by substruction (**1**) from (**2**) we obtain, respectively,
$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta \Leftrightarrow$
(**11**) $\cos\alpha\cos\beta = \frac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2}$ and
$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta \Leftrightarrow$
(**12**) $\sin\alpha\sin\beta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}$.
Similarly, (**3**) and (**4**) give us $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta \Leftrightarrow$
(**13**) $\sin\alpha\cos\beta = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2}$ and
(**14**) $\cos\alpha\sin\beta = \frac{\sin(\alpha + \beta) - \sin(\alpha - \beta)}{2}$.

9. Sum to Product Formulas.

Since for any α and β we have $\begin{cases} \alpha = x + y \\ \beta = x - y \end{cases} \Leftrightarrow \begin{cases} x = \frac{\alpha + \beta}{2} \\ y = \frac{\alpha - \beta}{2} \end{cases}$ then

$\cos\alpha + \cos\beta = \cos(x + y) + \cos(x - y) = 2\cos x\cos y = 2\cos\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}$
and $\cos\alpha - \cos\beta = \cos(x + y) - \cos(x - y) = -2\sin x\sin y = -2\sin\frac{\alpha + \beta}{2}\sin\frac{\alpha - \beta}{2}$.
Also we obtain $\sin\alpha + \sin\beta = \sin(x + y) + \sin(x - y) = 2\sin x\cos y = 2\sin\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}$

and $\sin\alpha - \sin\beta = \sin(x+y) - \sin(x-y) = 2\cos x \sin y = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$.
So we have more four formulas which give four standards of factorization.

(**15**) $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

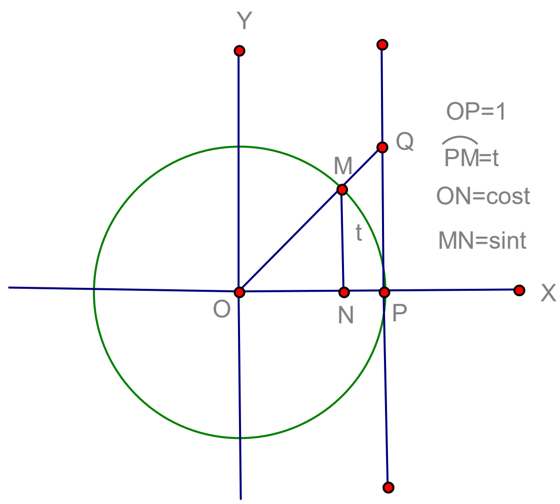
(**16**) $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

(**17**) $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

(**18**) $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$.

10. More trigonometric functions.

$\tan t := \frac{\sin t}{\cos t}$ (tangent of t)

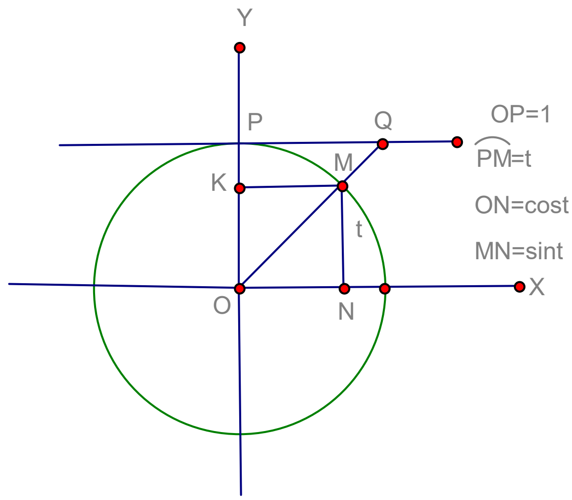


We call vertical line $\overleftrightarrow{QP}$ axes of tangents, because

$\frac{QP}{OP} = \frac{MN}{ON} \Leftrightarrow \frac{QP}{1} = \frac{\sin t}{\cos t} \Leftrightarrow QP = \tan t.$

$Dom(\tan t) = \mathbb{R} \setminus \{t \mid t \in \mathbb{R} \,\&\, \cos t = 0\} = \mathbb{R} \setminus \{\pi/2 + 2k\pi \mid k \in \mathbb{Z}\}.$

$\cot t := \frac{\cos t}{\sin t}$(cotangent of t)



We call horizontal line $\overleftrightarrow{QP}$ axes of cotangents, because

$\dfrac{QP}{OP} = \dfrac{OK}{KM} \Leftrightarrow \dfrac{QP}{1} = \dfrac{\cos t}{\sin t} \Leftrightarrow QP = \cot t.$

$Dom(\cot t) = \mathbb{R} \setminus \left\{t \mid t \in \mathbb{R} \,\&\, \sin t = 0\right\} = \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}.$

Also in trigonometry can be useful functions

$\sec t := \dfrac{1}{\cos t}$ with domain $\mathbb{R} \setminus \{\pi/2 + 2k\pi \mid k \in \mathbb{Z}\}$

and $\csc t := \dfrac{1}{\sin t}$ with domain $\mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$.

Both, tangent and cotangent are *odd* and *periodical* with period π.

Indeed, $\tan(-t) = \dfrac{\sin(-t)}{\cos(-t)} = \dfrac{-\sin t}{\cos t} = -\tan t$,

$\tan(t+\pi) = \dfrac{\sin(t+\pi)}{\cos(t+\pi)} = \dfrac{-\sin t}{-\cos t} = \tan t$ and $\cot(-t) = \dfrac{\cos(-t)}{\sin(-t)} = \dfrac{\cos t}{-\sin t} = -\cot t$,

$\cot(t+\pi) = \dfrac{\cos(t+\pi)}{\sin(t+\pi)} = \dfrac{-\cos t}{-\sin t} = -\cot t$

11. Identities with $\tan$ and $\cot$.

(19) $\tan t \cdot \cot t = 1.$

Since $\{k\pi \mid k \in \mathbb{Z}\} \cup \{\pi/2 + 2k\pi \mid k \in \mathbb{Z}\} =$, then

domain of identity is $\mathbb{R} \setminus \left\{\dfrac{k\pi}{2} \mid k \in \mathbb{Z}\right\}$.

(20) $1 + \tan^2 t = \dfrac{1}{\cos^2 t} = \sec^2 t;$

(21) $1 + \cot^2 t = \dfrac{1}{\sin^2 t} = \csc^2 t.$

12. Additional formulas for tangent and cotangent.

We have

$$\tan(\alpha+\beta) = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = \frac{\cos\alpha\cos\beta(\tan\alpha + \tan\beta)}{\cos\alpha\cos\beta(1 - \tan\alpha\tan\beta)} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

and $\cot(\alpha+\beta) = \dfrac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta} = \dfrac{\sin\alpha\sin\beta(\cot\alpha\cot\beta - 1)}{\sin\alpha\sin\beta(\cot\alpha + \cot\beta)} = \dfrac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$.

Thus, we have two more formulas:

(**22**) $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$. Domain $\alpha+\beta \neq \dfrac{\pi}{2}+k\pi, \alpha \neq \dfrac{\pi}{2}+m\pi, \beta \neq \dfrac{\pi}{2}+n\pi, k,m,n \in \mathbb{Z}$.

(**23**) $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$. Domain $\alpha+\beta \neq k\pi, \alpha \neq m\pi, \beta \neq n\pi, k,m,n \in \mathbb{Z}$.

In particular we obtain

(**24**) $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$ and

(**25**) $\cot 2\alpha = \dfrac{\cot^2\alpha-1}{2\cot\alpha}$.

13. All trig function via tangent of half argument.

By replacing α in formulas (**24**) and (**25**) with $\dfrac{\alpha}{2}$ we obtain

(**26**) $\tan\alpha = \dfrac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$ and

(**27**) $\cot\alpha = \dfrac{\cot^2\frac{\alpha}{2}-1}{2\cot\frac{\alpha}{2}} = \dfrac{1-\tan^2\frac{\alpha}{2}}{2\tan\frac{\alpha}{2}}$.

Also we have $\cos\alpha = \cos^2\dfrac{\alpha}{2}-\sin^2\dfrac{\alpha}{2} = \dfrac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}} = \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$

and $\sin\alpha = 2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2} = \dfrac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}} = \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$. So,

(**28**) $\cos\alpha = \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$

(**29**) $\sin\alpha = \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$.

We also should take in account following transformations:

(**30**) $\tan\alpha \pm \tan\beta = \dfrac{\sin\alpha}{\cos\alpha} \pm \dfrac{\sin\beta}{\cos\beta} = \dfrac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta} = \dfrac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$

(**31**) $\cot\alpha \pm \cot\beta = \dfrac{\cos\alpha}{\sin\alpha} \pm \dfrac{\cos\beta}{\sin\beta} = \dfrac{\sin\beta\cos\alpha \pm \cos\beta\sin\alpha}{\sin\alpha\sin\beta} = \dfrac{\sin(\beta\pm\alpha)}{\sin\alpha\sin\beta}$

(**32**) $\cot\alpha \pm \tan\beta = \dfrac{\cos\alpha}{\sin\alpha} \pm \dfrac{\sin\beta}{\cos\beta} = \dfrac{\cos(\alpha\mp\beta)}{\sin\alpha\cos\beta}$.

In particular, $\cot\alpha - \tan\alpha = 2\cot 2\alpha$.

14. Reduction formulas for tangent and cotangent.

(**33**) $\tan\left(\dfrac{\pi}{2}-\alpha\right) = \cot\alpha$ and $\cot\left(\dfrac{\pi}{2}-\alpha\right) = \tan\alpha$,

(**34**) $\tan\left(\dfrac{\pi}{2}+\alpha\right) = -\cot\alpha$ and $\cot\left(\dfrac{\pi}{2}+\alpha\right) = -\tan\alpha$.

15. Basic equations.

$\sin\alpha = \sin\beta \Leftrightarrow \sin\alpha - \sin\beta = 0 \Leftrightarrow 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = 0 \Leftrightarrow$

$$\left[\begin{array}{l} \cos\frac{\alpha+\beta}{2} = 0 \\ \sin\frac{\alpha-\beta}{2} = 0 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \frac{\alpha+\beta}{2} = \frac{\pi}{2} + k\pi \\ \frac{\alpha-\beta}{2} = mp \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \alpha = -\beta + (2k+1)\pi \\ \alpha = \beta + 2mp \end{array}\right. .$$

Both cases can be united in on formula

(**35**) $\sin\alpha = \sin\beta \Leftrightarrow \alpha = (-1)^n\beta + n\pi, n \in \mathbb{Z}$

$\cos\alpha = \cos\beta \Leftrightarrow \cos\alpha - \cos\beta = 0 \Leftrightarrow -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = 0 \Leftrightarrow$

$$\left[\begin{array}{l} \sin\frac{\alpha+\beta}{2} = 0 \\ \sin\frac{\alpha-\beta}{2} = 0 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \frac{\alpha+\beta}{2} = k\pi \\ \frac{\alpha-\beta}{2} = m\pi \end{array}\right. \Leftrightarrow \begin{array}{l} \alpha = -\beta + 2k\pi \\ \alpha = \beta + 2m\pi \end{array}$$. So,

(**36**) $\cos\alpha = \cos\beta \Leftrightarrow \alpha = \pm\beta + 2n\pi, n \in \mathbb{Z}$

$\tan\alpha = \tan\beta \Leftrightarrow \tan\alpha - \tan\beta = 0 \Leftrightarrow \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} = 0 \Leftrightarrow \alpha - \beta = k\pi \Leftrightarrow \alpha = \beta + k\pi, k \in \mathbb{Z}.$

(**37**) $\tan\alpha = \tan\beta \Leftrightarrow \alpha = \beta + k\pi, k \in \mathbb{Z}$

and similarly

(**38**) $\cot\alpha = \cot\beta \Leftrightarrow \alpha = \beta + k\pi, k \in \mathbb{Z}.$

Example of application.

Solve equation $\sin 2x = \frac{\sqrt{3}}{2}.$

Since $\frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$ then $\sin 2x = \frac{\sqrt{3}}{2} \Leftrightarrow \sin 2x = \sin\frac{\pi}{3} \Leftrightarrow 2x = (-1)^n\frac{\pi}{3} + n\pi \Leftrightarrow$
$x = (-1)^n\frac{\pi}{6} + \frac{n\pi}{2}.$

Problems.

Solve following equations:

$\sin\left(3x - \frac{\pi}{5}\right) = \frac{1}{2};\ \cos\left(\frac{\pi}{3} - 2x\right) = \frac{\sqrt{2}}{2};\ \tan 2x = 1.$

16. Amplituda formula.

First we will prove that if $\begin{cases} \sin\alpha = \sin\beta \\ \cos\alpha = \cos\beta \end{cases}$ and $\alpha, \beta \in [0, 2\pi)$ then $\alpha = \beta.$

Indeed, $\begin{cases} \sin\alpha = \sin\beta \\ \cos\alpha = \cos\beta \end{cases} \Leftrightarrow \begin{cases} \alpha = (-1)^n\beta + m\pi \\ \alpha = \pm\beta + 2n\pi \end{cases} \Rightarrow \pm\beta + 2n\pi = (-1)^n\beta + m\pi.$

Possible variants:

1. $\begin{cases} \alpha = \beta + 2k\pi \\ \alpha = \beta + 2n\pi \end{cases} \Leftrightarrow k = n = 0$ because $\alpha - \beta = 2k\pi \Rightarrow -2\pi < 2k\pi < 2\pi \Leftrightarrow$
   $k = 0 \Rightarrow \alpha = \beta.$

2. $\begin{cases} \alpha = \beta + 2k\pi \\ \alpha = -\beta + 2n\pi \end{cases} \Rightarrow \begin{cases} \alpha = (k+n)\pi \\ \beta = (n-k)\pi \end{cases} \Rightarrow \begin{cases} 0 \le n+k < 2 \\ 0 \le n-k < 2 \end{cases} \Leftrightarrow$

$$\begin{cases} 0 \le n+k \le 1 \\ 0 \le n-k \le 1 \end{cases} \Leftrightarrow \begin{cases} -k \le n \le 1-k \\ k \le n \le 1+k \end{cases} \Leftrightarrow \max\{k,-k\} \le n \le 1+\min\{k,-k\} \Leftrightarrow$$
$$|k| \le n \le 1-|k| \Rightarrow 2|k| \le 1 \Rightarrow k = 0 \Rightarrow \alpha = \beta.$$

3. $\begin{cases} \alpha = -\beta + (2k+1)\pi \\ \alpha = -\beta + 2n\pi \end{cases} \Rightarrow 2k+1 = 2n$ –the contradiction.

4. $\begin{cases} \alpha = -\beta + (2k+1)\pi \\ \alpha = \beta + 2n\pi \end{cases} \Rightarrow \begin{cases} 2\alpha = (2k+1+2n)\pi \\ 2\beta = (2k+1-2n)\pi \end{cases} \Rightarrow \begin{cases} 0 \le 2k+1+2n < 4 \\ 0 \le 2k+1-2n < 4 \end{cases} \Leftrightarrow$

$$\begin{cases} 0 \le 2k+2n \le 3 \\ 0 \le 2k-2n \le 3 \end{cases} \Leftrightarrow \begin{cases} 0 \le k+n \le 1 \\ 0 \le k-n \le 1 \end{cases} \Rightarrow 2|n| \le 1 \Rightarrow n = 0 \Rightarrow \alpha = \beta.$$

Let φ be some angle such that $\varphi \in [0, 2\pi)$ and $\cos\varphi = \dfrac{a}{\sqrt{a^2+b^2}}, \sin\varphi = \dfrac{b}{\sqrt{a^2+b^2}}$.

Then angle φ is determined uniquely and we obtain

$$a\sin x + b\cos x = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} \cdot \sin x + \frac{b}{\sqrt{a^2+b^2}} \cdot \cos x\right) =$$
$$\sqrt{a^2+b^2}\,(\sin x\cos\varphi + \cos x\sin\varphi) = \sqrt{a^2+b^2}\,\sin(x+\varphi).$$

Or, if φ is determined by $\cos\varphi = \dfrac{b}{\sqrt{a^2+b^2}}, \sin\varphi = -\dfrac{a}{\sqrt{a^2+b^2}}$ then

$$a\sin x + b\cos x = \sqrt{a^2+b^2}\,\cos(x+\varphi).$$

17. Sine and Cosine of the triple angle and, more general, of any multiple argument.

Applying sum to product formulas we obtain:

$\cos 3t + \cos t = 2\cos 2t\cos t \Leftrightarrow \cos 3t = 2(2\cos^2 t - 1)\cos t - \cos t \Leftrightarrow$

(**39**) $\cos 3t = \cos t(4\cos^2 t - 3)$

and

$\sin 3t + \sin t = 2\sin 2t\cos t \Leftrightarrow \sin 3t = 4\sin t\cos^2 t - \sin t \Leftrightarrow \sin 3t = \sin t(4\cos^2 t - 1) \Leftrightarrow$

(**40**) $\sin 3t = \sin t(3 - 4\sin^2 t)$.

For any natural n we have

$\cos(n+1)t + \cos(n-1)t = 2\cos nt \cdot \cos t \Leftrightarrow$

(**41**) $\cos(n+1)t = 2\cos nt \cdot \cos t - \cos(n-1)t, n \in \mathbb{N}$.

Using recurrence (**40**) we can recursively obtain $\cos 2t, \cos 3t, \cos 4t, \ldots$.

Similarly $\sin(n+1)t + \sin(n-1)t = 2\sin nt \cdot \cos t \Leftrightarrow$

(**42**) $\sin(n+1)t = 2\sin nt \cdot \cos t - \sin(n-1)t,\ n \in \mathbb{N}$.

We can see that in both cases $\cos nt$ and $\sin nt$ subject to the same recurrence

$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n \in \mathbb{N}$.

In the case of initial condition for $T_0(x) = 1, T_1(x) = x$ we obtain sequence of polynomials $(T_n(x))_{n\ge 0}$ such that $T_n(\cos t) = \cos nt$.

Such polynomials are called Chebishev's Polynomials of the first kind.

Example.

$\sin 4t = 2\sin 3t \cdot \cos t - \sin 2t = 2\sin t(3 - 4\sin^2 t) \cdot \cos t - 2\sin t \cos t =$

$2\cos t(3\sin t - 4\sin^3 t - \sin t) = 4\cos t(\sin t - 2\sin^3 t)$

(Another way: $\sin 4t = 2\sin 2t \cos 2t = 4\sin t \cos t(1 - 2\sin^2 t)$)

$\sin 5t = 2\sin 4t \cos t - \sin 3t \Leftrightarrow \sin 5t = 2 \cdot 4\cos t(\sin t - 2\sin^3 t) \cdot \cos t - \sin t(3 - 4\sin^2 t) \Leftrightarrow$

$\sin 5t = 8(1 - \sin^2 t)\sin t(1 - 2\sin^2 t) - \sin t(3 - 4\sin^2 t) =$

$\sin t(8(1 - \sin^2 t)(1 - 2\sin^2 t) - (3 - 4\sin^2 t)) = \sin t(16\sin^4 t - 20\sin^2 t + 5) =$

$\sin t\,(2\cos 2t + 2\cos 4t + 1)$.

Remark.

As application of triple-angle formula we will find $\sin \frac{\pi}{10}$.

Since $\sin \frac{\pi}{5} = \cos\left(\frac{\pi}{2} - \frac{\pi}{5}\right) = \cos \frac{3\pi}{10} \Leftrightarrow$

$2\sin \frac{\pi}{10} \cos \frac{\pi}{10} = \cos \frac{\pi}{10}\left(4\cos^2 \frac{\pi}{10} - 3\right) \Leftrightarrow 2\sin \frac{\pi}{10} = 4\cos^2 \frac{\pi}{10} - 3 \Leftrightarrow$

$2\sin \frac{\pi}{10} = 1 - 4\sin^2 \frac{\pi}{10}$ then denoting $x := \sin \frac{\pi}{10} > 0$ we will find value of $\sin \frac{\pi}{10}$ as positive root of quadratic equation $4x^2 + 2x - 1 = 0$, that is $x = \frac{-1 + \sqrt{5}}{4}$. Thus, $\sin \frac{\pi}{10} = \frac{-1 + \sqrt{5}}{4}$.

18. And more useful formulas.

1. $4\sin\left(\frac{\pi}{3} - \alpha\right)\sin\alpha \sin\left(\frac{\pi}{3} + \alpha\right) = \sin 3\alpha$;

2. $4\cos\left(\frac{\pi}{3} - \alpha\right)\cos\alpha \cos\left(\frac{\pi}{3} + \alpha\right) = \cos 3\alpha$;

3. $\tan\left(\frac{\pi}{3} - \alpha\right)\tan\alpha \tan\left(\frac{\pi}{3} + \alpha\right) = \tan 3\alpha$.

$$\tan\left(\frac{\pi}{3} - \alpha\right)\tan\alpha \tan\left(\frac{\pi}{3} + \alpha\right) = \frac{\sin\left(\frac{\pi}{3} - \alpha\right)\sin\alpha \sin\left(\frac{\pi}{3} + \alpha\right)}{\cos\left(\frac{\pi}{3} - \alpha\right)\cos\alpha \cos\left(\frac{\pi}{3} + \alpha\right)} =$$

$$\frac{\left(\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right)\sin\alpha\left(\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha\right)}{\left(\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha\right)\cos\alpha\left(\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha\right)} = \frac{(3\cos^2\alpha - \sin^2\alpha)\sin\alpha}{(\cos^2\alpha - 3\sin^2\alpha)\cos\alpha} =$$

$$\frac{(3 - 4\sin^2\alpha)\sin\alpha}{(4\cos^2\alpha - 3)\cos\alpha} = \frac{\sin 3\alpha}{\cos 3\alpha} = \tan 3\alpha.$$

Application.

Calculate without calculator and tables:

a) $\tan 20^\circ \tan 40^\circ \tan 80^\circ$.

b) $4(\cos 24^\circ + \cos 48^\circ - \cos 84^\circ - \cos 12^\circ)$

Solution.

a) For $\alpha = 20^\circ$ we have $\tan 40^\circ \tan 20^\circ \tan 80^\circ = \tan\left(\frac{\pi}{3} - \alpha\right)\tan\alpha \tan\left(\frac{\pi}{3} + \alpha\right) =$

$\tan 3\alpha = \tan 60^\circ = \sqrt{3}$.

b) Let $S := \cos 24^\circ + \cos 48^\circ - \cos 84^\circ - \cos 12^\circ = 2\cos 36^\circ \cos 12^\circ - 2\cos 48^\circ \cos 36^\circ =$

$2\cos 36^\circ(\cos 12^\circ - \cos 48^\circ) = 2\cos 36^\circ \cdot 2\sin 30^\circ \sin 18^\circ = 2\cos 36^\circ \sin 18^\circ =$
$2\cos 36^\circ \cos 72^\circ = \dfrac{2\cos 36^\circ \cos 72^\circ \sin 36^\circ}{\sin 36^\circ} = \dfrac{\cos 72^\circ \sin 72^\circ}{\sin 36^\circ} = \dfrac{\sin 144^\circ}{2\sin 36^\circ} =$
$\dfrac{\sin 36^\circ}{2\sin 36^\circ} = \dfrac{1}{2}$. Hence, $4(\cos 24^\circ + \cos 48^\circ - \cos 84^\circ - \cos 12^\circ) = 4S = 2.$

19. Sum to product with more addends.

$\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2} + 2\cos\left(\gamma + \dfrac{\alpha+\beta}{2}\right)\cos\dfrac{\alpha+\beta}{2} =$
$2\cos\dfrac{\alpha+\beta}{2}\left(\cos\dfrac{\alpha-\beta}{2} + \cos\left(\gamma+\dfrac{\alpha+\beta}{2}\right)\right) = 2\cos\dfrac{\alpha+\beta}{2} \cdot 2\cos\dfrac{\alpha+\gamma}{2}\cos\dfrac{\beta+\gamma}{2} \Leftrightarrow$

(**43**) $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\beta+\gamma}{2}\cos\dfrac{\gamma+\alpha}{2}.$

$\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2} - 2\cos\left(\gamma+\dfrac{\alpha+\beta}{2}\right)\sin\dfrac{\alpha+\beta}{2} =$
$2\sin\dfrac{\alpha+\beta}{2}\left(\cos\dfrac{\alpha-\beta}{2} - \cos\left(\gamma+\dfrac{\alpha+\beta}{2}\right)\right) = 2\sin\dfrac{\alpha+\beta}{2} \cdot 2\sin\dfrac{\alpha+\gamma}{2}\sin\dfrac{\beta+\gamma}{2} \Leftrightarrow$

(**44**) $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\beta+\gamma}{2}\sin\dfrac{\gamma+\alpha}{2}.$

20. Application to geometry:

In particular if α, β, γ be angles of a triangle then $\alpha+\beta+\gamma = \pi$ and
$\cos\alpha + \cos\beta + \cos\gamma = 1 + 4\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}$ and since $r = 4R\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}$
then $\cos\alpha + \cos\beta + \cos\gamma = 1 + \dfrac{r}{R}$.
Also, $\sin\alpha + \sin\beta + \sin\gamma = 4\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2} \Leftrightarrow s = 4R\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}$.

21. Exploring behavior of trigonometric functions.

Since $\sin t$ and $\cos t$ are 2π –periodical then suffice explore these function on interval with lengths of 2π.

1. $\sin t$ on $[-\pi,\pi]$ and $\cos$ on $[0,2\pi]$

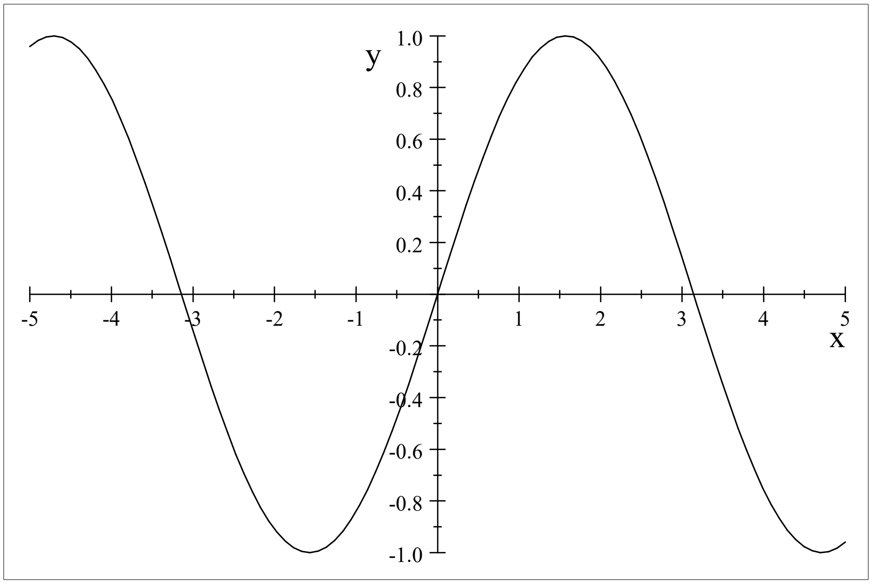
First note that since $\sin t$ is odd suffice explore behavior on $[0,\pi]$.

Part 1. $\sin t$ is strictly increasing on $[-\pi/2, \pi/2)$.

Indeed, let $-\pi/2 \le t_1 < t_2 < \pi/2$ then $\sin t_2 - \sin t_1 = 2\cos\dfrac{t_1+t_2}{2}\sin\dfrac{t_2-t_1}{2} > 0$
because $-\pi/2 \le \dfrac{t_1+t_2}{2} < \pi/2 \Rightarrow \cos\dfrac{t_1+t_2}{2} > 0$ and $0 \le \dfrac{t_2-t_1}{2} < \pi/2 \Rightarrow \sin\dfrac{t_2-t_1}{2} > 0$.
For $\pi/2 \le t_1 < t_2 \le \pi$ we have $0 < \pi - t_2 < \pi - t_1 \le \pi/2 \Rightarrow$
$\sin t_2 = \sin(\pi - t_2) < \sin(\pi - t_1) = \sin t_1$. Thus, $\sin t$ is decreasing on $[\pi/2, \pi]$.

Part 2. $\sin t$ is concave down on $[0,\pi]$.

Let $0 \le t_1 < t_2 \le \pi$. Then $\dfrac{\sin t_1 + \sin t_2}{2} \le \sin\dfrac{t_1+t_2}{2}$.
Indeed, $\dfrac{\sin t_1 + \sin t_2}{2} = \sin\dfrac{t_1+t_2}{2}\cos\dfrac{t_2-t_1}{2} \le \sin\dfrac{t_1+t_2}{2}$ because
$0 < \dfrac{t_1+t_2}{2} < \pi \Rightarrow \sin\dfrac{t_1+t_2}{2} > 0$ and $0 < \dfrac{t_2-t_1}{2} \le \pi/2 \Rightarrow 0 \le \cos\dfrac{t_2-t_1}{2} < 1$.
$\max\limits_{t\in[0,\pi]} \sin t = 1 = \sin\dfrac{\pi}{2}$.
Obtained information enough to draw graph of $\sin t$



Since $\sin\left(\frac{\pi}{2}+t\right) = \cos t$ then graph of $\cos t$ can be obtained by linear translation of graph $\sin t$ on $\frac{\pi}{2}$ left.

2. $\tan t$.

Since $\tan t$ is periodical with period π then suffice explore it's behavior on $(-\pi/2, \pi/2)$.

Part 1. $\tan t$ is increasing on $(-\pi/2, \pi/2)$.

Indeed, let $-\pi/2 < t_1 < t_2 < \pi/2$. Then $\tan t_2 - \tan t_1 = \dfrac{\sin(t_2 - t_1)}{\cos t_2 \cos t_1} > 0$

because $0 < t_2 - t_1 < \pi$.

Part 2. $\tan t$ is concave up on $[0, \pi/2)$ (concave down on $(-\pi/2, 0]$ because $\tan t$ is odd function).

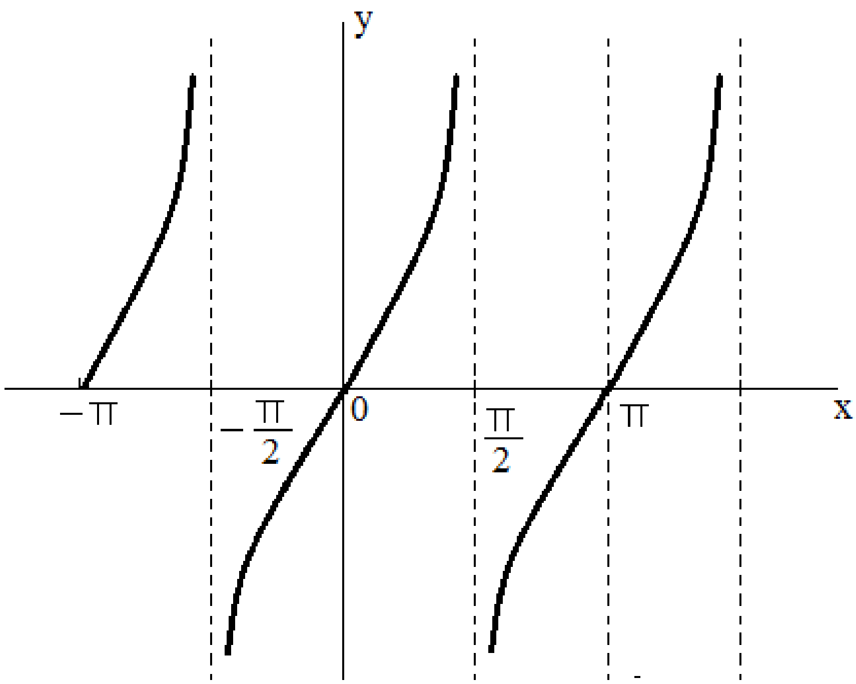
Let $0 \le t_1 < t_2 < \pi/2$. Then $\dfrac{\tan t_1 + \tan t_2}{2} \ge \tan \dfrac{t_1 + t_2}{2}$.

Indeed, $\dfrac{\tan t_1 + \tan t_2}{2} \ge \tan \dfrac{t_1 + t_2}{2} \iff \dfrac{\sin(t_1 + t_2)}{\cos t_1 \cos t_2} \ge \dfrac{2\sin\frac{t_1+t_2}{2}}{\cos\frac{t_1+t_2}{2}} \iff$

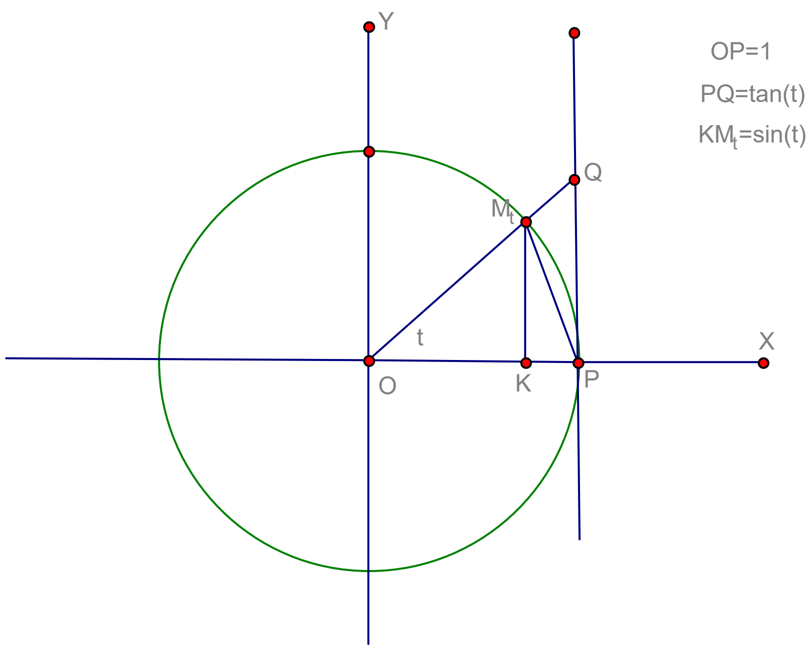
$\dfrac{\cos\frac{t_1+t_2}{2}}{\cos t_1 \cos t_2} \ge \dfrac{1}{\cos\frac{t_1+t_2}{2}} \iff \cos^2\dfrac{t_1+t_2}{2} \ge \cos t_1 \cos t_2 \iff$

$2\cos^2\dfrac{t_1+t_2}{2} \ge 2\cos t_1 \cos t_2 \iff 1 + \cos(t_1 + t_2) \ge 2\cos t_1 \cos t_2 \iff$

$1 \ge \cos(t_2 - t_1)$.



21. Basic trigonometric Inequalities.



First we recall that area of the sector that organized by angle t (in radian measure) in the circle of radius R equal $\frac{tR^2}{2}$.

Let $t \in (0, \pi/2)$ and let sM_tOP be sector of the unite circle.

Then $Area(\triangle M_tOP) < Area(sM_tOP) < Area(\triangle QOP) \Leftrightarrow$

$$\frac{M_tP \bullet OP}{2} < Area(sM_tOP) < \frac{QP \bullet OP}{2} \Leftrightarrow \frac{\sin t \bullet 1}{2} < \frac{t \bullet 1^2}{2} < \frac{\tan t \bullet 1}{2} \Leftrightarrow$$

(!) $\sin t < t < \tan t$

And since $t < \tan t \Leftrightarrow t\cos t < \sin t$ we obtain one more inequality

(!!) $t\cos t < \sin t < t.$

Let $t \in (-\pi/2, \pi/2) \setminus \{0\}$. Since $\sin t$ is odd then $\frac{\sin t}{t}$ is even function and then $\frac{\sin t}{t} = \frac{\sin|t|}{|t|}$.

Also since $\cos t$ is even then $\cos t = \cos|t|$ and we have for any $t \in (-\pi/2, \pi/2) \setminus \{0\}$

$$|t|\cos|t| < \sin|t| < |t| \Leftrightarrow \cos|t| < \frac{\sin|t|}{|t|} < 1 \Leftrightarrow$$

(!!!) $\cos t < \frac{\sin t}{t} < 1, t \in (-\pi/2, \pi/2) \setminus \{0\}.$

Inequalities (!), (!!), (!!!) are very important for calculus.

Problem.

Prove that $1 - \frac{t^2}{2} < \cos t$ and $\sin t < t - \frac{t^3}{4}, t \in (0, \pi/2).$

Several applications.

1. We will prove that function $\frac{\sin t}{t}$ is decreasing on $(0, \pi/2)$.

Let $t \in (0, \pi/2)$ and $h > 0$ such that $t + h \in (0, \pi/2)$ then $\frac{\sin t}{t} - \frac{\sin(t+h)}{t+h} =$

$$\frac{(t+h)\sin t - t\sin(t+h)}{t(t+h)} = \frac{t\sin t + h\sin t - t\sin t\cos h - t\cos t\sin h}{t(t+h)} =$$

$$\frac{t\sin t(1-\cos h) + h\sin t - t\cos t\sin h}{t(t+h)} =$$

$$\frac{t\sin t(1-\cos h) + (h\sin t - ht\cos t) + (ht\cos t - t\cos t\sin h)}{t(t+h)} =$$

$$\frac{2t\sin t\sin^2\frac{h}{2} + h\cos t(\tan t - t) + t\cos t(h - \sin h)}{t(t+h)} > 0$$

because $h > \sin h$ and $\tan t > t$.

In particular, since $\frac{\sin t}{t}$ decreasing then $n\sin\frac{t}{n}$ increasing in $n \in \mathbb{N}$,

that is $n\sin\frac{t}{n} < (n+1)\sin\frac{t}{n+1}, n \in \mathbb{N}, t \in (0, \pi/2).$

Indeed, $n\sin\frac{t}{n} < (n+1)\sin\frac{t}{n+1} \Leftrightarrow \frac{\sin\frac{t}{n}}{\frac{t}{n}} < \frac{\sin\frac{t}{n+1}}{\frac{t}{n+1}}$ because $\frac{t}{n} > \frac{t}{n+1}$.

2. We will prove that function $\frac{\tan t}{t}$ is increasing on $(0, \pi/2)$, $0 < x < \pi/2, h > 0$ and $x + h < \pi/2.$

$$\frac{\tan(x+h)}{x+h} - \frac{\tan x}{x} = \frac{\sin(x+h)}{(x+h)\cos(x+h)} - \frac{\sin x}{x\cos x} = \frac{x\cos x\sin(x+h) - (x+h)\sin x\cos(x+h)}{(x+h)x\cos x\cos(x+h)}$$

$x\cos x\sin(x+h) - (x+h)\sin x\cos(x+h) =$

$x\cos x\sin x\cos h + x\cos^2 x\sin h - (x+h)\sin x\cos x\cos h + (x+h)\sin^2 x\sin h =$

$x\cos x\sin x\cos h + \underline{x\cos^2 x\sin h} - (x+h)\sin x\cos x\cos h + \underline{x\sin^2 x\sin h} + h\sin^2 x\sin h =$

$\underline{x\sin h} - h\sin x\cos x\cos h + h\sin^2 x\sin h > h(x - \sin x) + h\sin^2 x\sin h > 0.$

3. Proof that $2x - \tan x \uparrow x \in (0, \pi/4)$ (without derivatives).
Let $g(x) := 2x - \tan x$ and $h > 0$ such that $x + h < \pi/4$.
Then $g(x+h) - g(x) = 2h - \tan(x+h) + \tan x = 2h - \dfrac{\sin h}{\cos(x+h)\cos x}$.
Since $\cos(x+h) < \cos x$ and $\dfrac{\sin h}{h} < 1$ then $2h - \dfrac{\sin h}{\cos(x+h)\cos x} > 2h - \dfrac{\sin h}{\cos^2(x+h)} =$
$h\left(2 - \dfrac{\frac{\sin h}{h}}{\cos^2(x+h)}\right) > h\left(2 - \dfrac{1}{\cos^2(x+h)}\right) = h(1 - \tan^2(x+h)) \geq 0.$ ■

4. Proof that $\dfrac{\sin^2 x}{x} \uparrow x \in (0, \pi/4)$ without derivatives.
Let $h > 0$ such that $x + h < \pi/4$.
Note that $\dfrac{\sin^2(x+h)}{x+h} - \dfrac{\sin^2 x}{x} = \dfrac{x\sin^2(x+h) - (x+h)\sin^2 x}{x(h+x)}$.
And we have
$x\sin^2(x+h) - (x+h)\sin^2 x = x(\sin^2(x+h) - \sin^2 x) - h\sin^2 x =$
$x\sin(2x+h)\sin h - h\sin^2 x > x\sin(2x+h)h\cos h - h\sin^2 x = h(x\sin(2x+h)\cos h - \sin^2 x) =$
$h\left(x\dfrac{\sin(2x+2h) + \sin 2x}{2} - \sin^2 x\right) > h(x\sin 2x - \sin^2 x) = h\sin x\cos x(2x - \tan x) > 0.$

5. Proof that $x + \cos x \uparrow x \in \mathbb{R}$.
Let $g(x) := x + \cos x, x \in \mathbb{R}$ and $0 < h < \pi/2$.
Then $g(x+h) - g(x) = h + \cos(x+h) - \cos x = h - 2\sin(x + h/2)\sin(h/2) \geq h - 2\sin(h/2) =$
$2(h/2 - \sin(h/2)) > 0.$

22. Inverse trigonometric functions.

1. **Inverse for** $\sin t$.
Since $\sin t$ is strictly increasing function on $[-\pi/2, \pi/2]$ such that $\sin([-\pi/2, \pi/2]) = [-1, 1]$ and $[-1, 1] = range_{\mathbb{R}}(\sin)$ then restriction of mapping $\sin : \mathbb{R} \to [-1, 1]$ on the segment $[-\pi/2, \pi/2]$, that is mapping $\sin : [-\pi/2, \pi/2] \longrightarrow [-1, 1]$ is bijection Therefore, defined inverse mapping $\sin^{-1} : [-1, 1] \longrightarrow [-\pi/2, \pi/2]$ which for any real number $s \in [-1, 1]$ set in correspondence angle (real number) $t \in [-\pi/2, \pi/2]$ such that

$$\begin{cases} \sin^{-1}(\sin t) = t, \quad t \in [-\pi/2, \pi/2] \\ \sin(\sin^{-1}(s)) = s, \; s \in [-1, 1] \end{cases}.$$

2. **Inverse for** $\cos t$.
Since $\cos s$ is strictly decreasing function on $[0, \pi]$ such that $\cos([0, \pi]) = [-1, 1]$ and $[-1, 1] = range_{\mathbb{R}}(\cos t)$ then restriction of mapping $\cos : \mathbb{R} \to [-1, 1]$ on the segment $[0, \pi]$, that is mapping $\cos : [0, \pi] \longrightarrow [-1, 1]$ is bijection Therefore, defined inverse mapping $\cos^{-1} : [-1, 1] \longrightarrow [0, \pi]$ which for any real number $s \in [-1, 1]$ set in correspondence angle (real number) $t \in [0, \pi]$ such that

$$\begin{cases} \cos^{-1}(\cos t) = t, \quad t \in [0, \pi] \\ \cos(\cos^{-1}(s)) = s, \; s \in [-1, 1] \end{cases}.$$

3. **Inverse for** $\tan t$.
Since $\tan t$ is strictly increasing function on $(-\pi/2, \pi/2)$ such that $\tan((-\pi/2, \pi/2)) = (-\infty, \infty)$

and $(-\infty,\infty) = range_{\mathbb{R}}(\tan)$ then restriction of mapping $\tan\ : Dom(\tan) \to (-\infty,\infty)$ on the interval $(-\pi/2,\pi/2)$, that is mapping $\tan\ : (-\pi/2,\pi/2) \longrightarrow (-\infty,\infty)$ is bijection
Therefore, defined inverse mapping $\tan^{-1}\ : (-\infty,\infty) \longrightarrow (-\pi/2,\pi/2)$ which for any real number $s \in (-\infty,\infty)$ set in correspondence angle (real number) $t \in (-\pi/2,\pi/2)$ such that

$$\begin{cases} \tan^{-1}(\tan t) = t,\ \ t \in (-\pi/2,\pi/2) \\ \tan(\tan^{-1}(s)) = s,\ s \in (-\infty,\infty) \end{cases}.$$

4. **Inverse for** $\cot t$.

Since $\cot t$ is strictly decreasing function on $(0,\pi)$ such that $\tan((0,\pi)) = (-\infty,\infty)$ and $(-\infty,\infty) = range_{\mathbb{R}}(\cot)$ then restriction of mapping $\cot\ : Dom(\tan) \to (-\infty,\infty)$ on the interval $(0,\pi)$, that is mapping $\cot\ : (0,\pi) \longrightarrow (-\infty,\infty)$ is bijection
Therefore, defined inverse mapping $\cot^{-1}\ : (-\infty,\infty) \longrightarrow (0,\pi)$ which for any real number $s \in (-\infty,\infty)$ set in correspondence angle (real number) $t \in (0,\pi)$ such that

$$\begin{cases} \cot^{-1}(\cot t) = t,\ \ t \in (0,\pi) \\ \cot(\cot^{-1}(s)) = s,\ s \in (-\infty,\infty) \end{cases}.$$

Properties and formulas.

Prove that.

1. $\sin^{-1}(-x) = -\sin^{-1}(x),\ x \in [-1,1];$
2. $\cos^{-1}(-x) = \pi - \cos^{-1}(x),\ x \in [-1,1];$
3. $\tan^{-1}(-x) = -\tan^{-1}x,\ x \in \mathbb{R};$
4. $\cot^{-1}(-x) = \pi - \cot^{-1}(x),\ x \in \mathbb{R};$
5. $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2},\ x \in [-1,1];$
6. $\tan^{-1}(x) + \cot^{-1}(x) = \frac{\pi}{2},\ x \in \mathbb{R};$
7. $\sin^{-1}(x) = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right), x \in (0,1).$

to be continued....